

# Negative Energies on the Brane

Dan N. Vollick

Department of Mathematics and Statistics

and

Department of Physics

Okanagan University College

3333 College Way

Kelowna, B.C.

V1V 1V7

## Abstract

It has recently been proposed that our universe is a three-brane embedded in a higher dimensional spacetime. Here I show that black holes on the brane, black strings intersecting the brane, and gravitational waves propagating in the bulk induce an effective energy-momentum tensor on the brane that contains negative energy densities.

# Introduction

It has recently been suggested that some of the extra dimensions required by string theory may be “large” [1, 2] or even infinite [3]. In the scenario proposed in [1, 2] the spacetime is  $M^{(4)} \times K$ , where  $M^{(4)}$  is four dimensional Minkowski space and  $K$  is a compact manifold. The size of the extra dimensions must be  $\lesssim 5 \times 10^{-5}$  mm to be consistent with observations [4, 5]. In the Randall and Sundrum model [3] our three-brane is a domain wall separating two semi-infinite anti-de Sitter regions. In both scenarios the standard model fields are confined to the brane and gravity propagates in the bulk.

The Einstein field equations on the brane were derived by Shiromizu, Maeda, and Sasaki [6]. The effective four dimensional energy-momentum tensor contains terms involving surface stresses on the brane and a term involving the five dimensional Weyl tensor evaluated at the brane. In the absence of surface stresses the energy-momentum tensor on the brane reduces to the Weyl term which does not necessarily satisfy the weak, dominant, or strong energy conditions. Thus, in these “brane-world” scenarios the effective four dimensional energy-momentum tensor may contain negative energy densities.

The existence of negative energy densities in quantum field theory has been known for a long time [7]. However, it has been shown by Ford and Roman (see [8, 9] for a small subset of their papers) that negative energy densities in Minkowski space must satisfy quantum inequalities. For electromagnetic and scalar fields in a four dimensional spacetime these inequalities state that an observer can measure a negative energy density for a maximum time  $t \sim |\rho|^{-1/4}$ . Thus the more negative the energy density the shorter the time that it can persist. Quantum inequalities therefore put constraints on the existence of negative energies in quantum field theories in flat spacetime. These constraints are important otherwise violations of causality, cosmic censorship, and the second law of thermodynamics could be produced.

In this paper I will show that negative energy densities are easily produced on a three-brane embedded in a higher dimensional space. For simplicity I will take the surface stresses on the brane to vanish and the negative energies will originate in the Weyl part of the effective four dimensional energy-momentum tensor. These negative energies do not in general satisfy a quantum inequality and could therefore present observational problems for the “brane world” scenarios.

## Negative Energy Densities on a Three-Brane

First consider a black hole of radius  $r_0$  on a brane embedded in a  $d$ -dimensional spacetime ( $d > 4$ ). Let the size of the extra dimensions be  $L$  and let  $L \gg r_0$ . The  $d$ -dimensional metric near the black hole is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{(d-2)}^2 \quad (1)$$

where

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^{(d-3)} \quad (2)$$

and the induced metric on the brane is

$$ds_{(4)}^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{(2)}^2. \quad (3)$$

This is not a four dimensional Schwarzschild spacetime but it can be interpreted as a spacetime containing a black hole and additional matter.

The effective four dimensional energy-momentum tensor is

$$T_{\mu\nu}^{(4)} = -\frac{1}{8\pi} G_{\mu\nu}^{(4)} \quad (4)$$

and its non-zero components are

$$T_{tt}^{(4)} = -\frac{(d-4)}{8\pi r^2} \left(\frac{r_0}{r}\right)^{d-3} \left[1 - \left(\frac{r_0}{r}\right)^{d-3}\right], \quad (5)$$

$$T_{rr}^{(4)} = \frac{(d-4)}{8\pi r^2} \left(\frac{r_0}{r}\right)^{d-3} \left[1 - \left(\frac{r_0}{r}\right)^{d-3}\right]^{-1}, \quad (6)$$

$$T_{\theta\theta}^{(4)} = -\frac{(d-4)(d-3)}{16\pi} \left(\frac{r_0}{r}\right)^{d-3}, \quad (7)$$

and  $T_{\phi\phi}^{(4)} = T_{\theta\theta}^{(4)} \sin^2(\theta)$ . The energy density  $\rho = -T_t^t$  is

$$\rho = -\frac{(d-4)}{8\pi r^2} \left(\frac{r_0}{r}\right)^{d-3} \quad (8)$$

and the brane therefore contains a negative energy density. It is important to remember that (5) to (8) are valid only for  $r \ll L$ . For  $r \gg L$  the spacetime becomes approximately Schwarzschild and  $f(r) \simeq 1 - 2m/r$  where  $m$  is the four dimensional asymptotic mass. It was shown in [10] that this mass is the same as the mass measured in the bulk. The negative mass contained within a radius  $0 < R \ll L$  can be found from [11]

$$m(R) = 4\pi \int_0^R r^2 \rho(r) dr \quad (9)$$

which diverges. Thus there is an infinite amount of negative mass in the spacetime.

In five dimensions the above energy-momentum tensor can also be found using the approach of Shiromizu et. al. [6]. Using the Gauss-Codacci equations they show that in the absence of surface stresses and a cosmological constant the Einstein tensor is given by

$$G_{\mu\nu}^{(4)} = -C_{\alpha\beta\rho\sigma}^{(5)} \eta^\alpha \eta^\rho q_\mu^\beta q_\nu^\sigma \quad (10)$$

where  $C_{\alpha\beta\rho\sigma}^{(5)}$  is the five dimensional Weyl tensor evaluated at the brane,  $\eta^\alpha$  is a unit vector orthogonal to the brane, and  $q_{\mu\nu} = g_{\mu\nu} - \eta_\mu\eta_\nu$  is the induced metric on the brane. The five dimensional metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2[d\chi^2 + \sin^2\chi d\Omega_{(2)}^2] \quad (11)$$

and the brane is located at  $\chi = \pi/2$ . The energy-momentum tensor on the brane is

$$T_{\mu\nu}^{(4)} = \frac{1}{8\pi r^2} C_{5\mu 5\nu}^{(5)} \quad (12)$$

and the relevant non-zero components of the Weyl tensor at  $\chi = \pi/2$  are

$$C_{5151}^{(5)} = -\left(\frac{r_0}{r}\right)^2 \left[1 - \left(\frac{r_0}{r}\right)^2\right], \quad (13)$$

$$C_{5252}^{(5)} = \left(\frac{r_0}{r}\right)^2 \left[1 - \left(\frac{r_0}{r}\right)^2\right]^{-1}, \quad (14)$$

$$C_{5353}^{(5)} = -r_0^2, \quad (15)$$

and

$$C_{5454}^{(5)} = -r_0^2 \sin^2(\theta). \quad (16)$$

Substituting these into (4) and (10) gives the same energy-momentum tensor as found above.

A neutral black string wrapped around the extra dimension can also produce negative energy densities on the brane. Such a black string in a  $d$ -dimensional spacetime can be obtained by taking the product of  $S^1$  with a  $(d-1)$ -dimensional Schwarzschild spacetime. The induced metric on the brane near the string will be

$$ds_{(4)}^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_{(2)}^2 \quad (17)$$

where

$$g(r) = 1 - \left(\frac{r_0}{r}\right)^{d-4}. \quad (18)$$

and  $r_0 \ll L$ . Thus, if  $d > 5$  the brane will contain negative energy densities.

Negative energies can also be produced on the brane by a gravitational wave propagating in the bulk. Consider the five dimensional spacetime

$$ds^2 = -dt^2 + [1 + \epsilon \cos k(t-z)]dx^2 + dy^2 + dz^2 + 2\epsilon \cos k(t-z)dx dw + [1 - \epsilon \cos k(t-z)]dw^2 \quad (19)$$

that contains a linearized gravitational wave propagating in the  $z$ -direction. The fifth dimension is topologically  $S^1$  and is labelled by  $w$ . The brane is located at  $w = 0$  and the effective four dimensional energy-momentum tensor, to lowest order in  $\epsilon$ , is

$$T_{tt}^{(4)} = \frac{\epsilon k^2}{16\pi} \cos k(t - z), \quad (20)$$

$$T_{xx}^{(4)} = T_{yy}^{(4)} = 0, \quad (21)$$

$$T_{zz}^{(4)} = \frac{\epsilon k^2}{16\pi} \cos k(t - z), \quad (22)$$

and

$$T_{tz}^{(4)} = T_{zt}^{(4)} = -\frac{\epsilon k^2}{16\pi} \cos k(t - z). \quad (23)$$

The energy density on the brane, to lowest order in  $\epsilon$ , is

$$\rho = \frac{\epsilon k^2}{16\pi} \cos k(t - z) \quad (24)$$

and consists of regions of positive and negative energy densities propagating at the speed of light. Note that  $\rho$  is proportional to  $\epsilon$  in contrast to the (positive) energy density in a four dimensional gravitational wave, which is proportional to  $\epsilon^2$ . The same energy-momentum tensor can be obtained using (4) and (10).

Finally, consider a gravitational wave incident upon the brane. The metric will be taken to be

$$ds^2 = -dt^2 + [1 + \epsilon \cos k(t - w)]dx^2 + [1 - \epsilon \cos k(t - w)]dy^2 + 2\epsilon \cos k(t - w)dx dy + dz^2 + dw^2. \quad (25)$$

For the spacetime to be periodic in the fifth dimension  $k$  must satisfy  $k = 2n\pi/L$ , where  $n$  is an integer. The effective four dimensional energy-momentum tensor, to lowest order in  $\epsilon$ , is

$$T_{tt}^{(4)} = T_{zz}^{(4)} = 0, \quad (26)$$

$$T_{xx}^{(4)} = -\frac{\epsilon k^2}{16\pi} \cos(kt), \quad (27)$$

$$T_{yy}^{(4)} = \frac{\epsilon k^2}{16\pi} \cos(kt), \quad (28)$$

$$T_{xy}^{(4)} = T_{yx}^{(4)} = -\frac{\epsilon k^2}{16\pi} \cos(kt). \quad (29)$$

This energy-momentum tensor violates the weak, strong, and dominant energy conditions. Thus, even though  $\rho = 0$  in these coordinates some observers on the brane will measure negative energy densities.

## Conclusion

I have shown that black holes on the brane, black strings intersecting the brane, and gravitational waves propagating in the bulk produce negative energy densities on the brane. The total negative energy around the black hole and black string diverges ( $d > 5$  for the string) and the negative energy density produced by a gravitational wave propagating in the bulk is proportional to the wave amplitude, not the amplitude squared.

## References

- [1] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. **B429**, 263 (1998), hep-ph/9803315
- [2] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. **B436**, 257 (1998), hep-ph/9804398
- [3] L. Randall and R. Sundrum, hep-th/9906064
- [4] S. Cullen and M. Perelstein, Phys. Rev. Lett. **83**, 268 (1999), hep-ph/9903422
- [5] L.J. Hall and D. Smith, Phys. Rev. **D60**, 085008 (1999), hep-ph/9904267
- [6] T. Shiromizu, K. Maeda, and M. Sasaki, gr-qc/9910076
- [7] H. Epstein, V. Glaser, and A. Jaffe, Nuovo Cimento **36**, 1016 (1965)
- [8] L.H. Ford and T.A. Roman, Phys. Rev **D55**, 2082 (1997)
- [9] L.H. Ford, Phys. Rev. **D43**, 3972 (1991)
- [10] R. Emparan, G.T. Horowitz, R.C. Myers, hep-th/0003118
- [11] C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation (San Francisco: Freeman) 1973